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A metasemantic challenge for mathematical determinacy

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Abstract This paper investigates the determinacy of mathematics. We begin by clarifying how we are understanding the notion of determinacy (Sect. 1) before turning to the questions of whether and how famous independence results bear on issues of determinacy in mathematics (Sect. 2). From there, we pose a metasemantic challenge for those who believe that mathematical language is determinate (Sect. 3), motivate two important constraints on attempts to meet our challenge (Sect. 4), and then use these constraints to develop an argument against determinacy (Sect. 5) and discuss a particularly popular approach to resolving indeterminacy (Sect. 6), before offering some brief closing reflections (Sect. 7). We believe our discussion poses a serious challenge for most philosophical theories of mathematics, since it puts considerable pressure on all views that accept a non-trivial amount of determinacy for even basic arithmetic.

Keywords Determinacy · Indeterminacy · Metasemantics · Philosophy of mathematics · Incompleteness

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1 Independence and indeterminacy

A sentence ϕ in the language of a theory T is *independent* of T just in case neither ϕ nor its negation is provable from T.[1] At least since Euclid's *Elements*, mathematics has been presented by way of axiomatic theories and mathematicians have wondered whether certain mathematical sentences were independent of particular mathematical theories. The failure of strenuous efforts to derive Euclid's fifth postulate from the other four eventually led to a proof that the fifth postulate is independent of the other four.[2]

Contemporary mathematics is replete with independence results: Gödel and Rosser sentences, canonical consistency sentences, Goodstein's theorem, the Continuum Hypothesis, Souslin's problem, etc.[3] Many philosophers and mathematicians have taken at least some of these results to bear, somehow, someway, on the thesis that mathematics is *indeterminate*. Here we will investigate the issue of determinacy. In addition to the general issue, we are particularly interested in whether and how independence results bear on the question of mathematical indeterminacy.

Before investigating these questions, we need to clarify our terminology. As a first step in this direction, note that in some discussions the terms "pluralism" and "indeterminacy" are used for similar or intimately related ideas. Obviously, philosophers can use terminology however they choose as long as they're explicit about it, but we think that "mathematical pluralism" and "mathematical indeterminacy" are most naturally used for two distinct notions, given the associations of the relevant terms. We'll understand "pluralism" in the following sense:

> PLURALISM: There are apparently incompatible theories of subject D that are all equally correct

So a pluralist about set theory might maintain that well-founded set theories and non-well-founded set theories can both be equally correct, even though they seem to directly contradict each other. Making this perfectly clear would involve saying much more about what is meant by "apparently incompatible", "a theory of subject D", and by "equally correct". We think that this can and should be done, but we won't attempt to do so here.[4] Our main concern is not with pluralism but rather with indeterminacy, which we'll understand in the following sense:

> INDETERMINACY: At least some D-claims do not have a determinate truth value, that is, some D-claims are neither determinately true nor determinately false

This notion has been much used but somewhat underdiscussed, at least in the philosophy of mathematics, but similar notions are familiar from discussions of vagueness and the semantic paradoxes.

[1] Throughout we assume classical logic—an untendentious assumption in mainstream mathematics—for full generality we would need to distinguish between, for different logics L, different notions of L-independence.

[2] From a modern point of view, the first clear independence proof is found in Beltrami (1868a, b).

[3] See Gödel (1931), Rosser (1936), Goodstein (1944), Kirby and Paris (1982), Gödel (1939), Cohen (1963, 1964), Suslin (1920), Jech (1967), Tennenbaum (1968), and Solovay and Tennenbaum (1971).

[4] See Sect. 4 of Warren (2015) for a relevant general characterization of conceptual pluralism.

Unlike some notions of determinacy that are discussed in philosophy, the notion as we're understanding it here is *semantic* or *conceptual*—it concerns the content of our mathematical notions. In saying this, we mainly mean to be ruling out the possibility that the relevant notion of indeterminacy is merely epistemic—an expression of ineradicable ignorance or the like.[5] To illustrate: claiming that the continuum hypothesis (*CH*) is indeterminate in this sense is incompatible with claiming that there is a *fact of the matter* about whether or not $\aleph_1 = 2^{\aleph_0}$ that we are incapable of discovering. Accordingly, we can informally gloss the claim that *CH* is indeterminate as the claim that there is no fact of the matter as to whether or not *CH* is true.

On this notion of determinacy, determinate truth is roughly equivalent to "truth on any admissible interpretation of the language" where, as is usual, an interpretation assigns semantic values to all expressions of the language. To say that an interpretation is "admissible" is to say that it is compatible with the meanings of expressions in the relevant fragment of the language.[6] Much more could be said about the notion of determinacy, but the gloss we have provided is enough for our purposes.

There may be interesting connections between the issues of pluralism and determinacy, but there is no obvious entailment either way. Determinacy does not obviously entail non-pluralism. For instance, suppose that our actual logical terminology is classical and fully determinate, in the sense that every claim about entailment has a determinate truth value. This does not rule out the possibility of an intuitionistic logical practice that is equally correct from an objective point of view. On the other hand, indeterminacy does not obviously entail pluralism. Suppose that vagueness is a species of indeterminacy, and that vagueness is essential to certain subject-matters. From this it does not follow that there are incompatible but equally correct theories of such subject-matters. Our focus here will be primarily on indeterminacy, rather than pluralism.

2 From independence to indeterminacy?

Folkloric arguments for mathematical indeterminacy often give pride of place to independence results. Here is a simple but diagnostically useful example of an argument from independence to indeterminacy, using the continuum hypothesis (*CH*) and Zermelo–Fraenkel set theory with the axiom of choice (*ZFC*):

A1 *CH* is independent of *ZFC*
A2 A sentence in the language of set theory is indeterminate if and only if it is independent of *ZFC*
A3 *So*: *CH* is indeterminate (1,2)

If *ZFC* is consistent, then the truth of premise A1 follows from the work of Gödel and Cohen.[7] Premise A2 expresses a simple—many would say simplistic—picture of set-theoretic truth, according to which a sentence in the language of set theory is

[5] Williamson (1994) sees the indeterminacy induced by vagueness as being merely epistemic in this sense.

[6] This is roughly equivalent to notions of "super-truth" as used in supervaluationist treatments of vagueness.

[7] Of course, if *ZFC* is inconsistent then no sentence in the language of set theory, including both *CH* and its negation, is independent of the theory.

determinately true just in case it is provable from a distinguished, axiomatic theory of sets, in this case, *ZFC*.

While it has rarely if ever been presented so starkly, we believe that some version of this argument is taken seriously by many mathematicians and philosophers of mathematics.[8] But consider now a directly analogous argument from independence to indeterminacy, this time using the Rosser sentence R for Peano arithmetic (*PA*):[9]

B1	R is independent of *PA*
B2	A sentence in the language of arithmetic is indeterminate if and only if it is independent of *PA*
B3	*So*: R is indeterminate (1,2)

If *PA* is consistent, then the truth of premise B1 follows from the work of Rosser and Gödel. And premise B2 expresses a simple—many would say simplistic—picture of arithmetical truth, according to which a sentence in the language of arithmetic is determinately true just in case it is provable from a distinguished, axiomatic theory of arithmetic, in this case, *PA*.

Despite the evident similarities between the A-argument and the B-argument and the seriousness with which the A-argument is treated, in our experience, the B-argument is not taken seriously as an argument for the indeterminacy of arithmetic.[10] This despite the fact that premise B2 expresses an account of arithmetical truth directly analogous to premise A2's account of set-theoretic truth. Both set-theoretic and arithmetical truth, on these accounts, are at least extensionally equivalent to provability in a particular, distinguished mathematical theory: a set-theoretic [arithmetical] sentence ϕ is true just in case ϕ is provable from *ZFC* [*PA*]. Given this, what explains why the A-argument is typically treated seriously while the B-argument is dismissed out of hand?

At this point, some readers will be impatiently noting that there are relevant disanalogies between the examples on which these two arguments are based. For one thing, many think that there is independent reason to think that *PA*'s Rosser sentence is *true*.[11] For another, while *CH* was a clearly formulated mathematical problem that was only later found to be independent of standard set theory, the Rosser sentence R is a completely *ad hoc*, cooked up arithmetical claim that is, in effect, *designed* to be undecidable. In fact, the work of Gödel and Rosser provides us with a recipe for constructing undecidable sentences from consistent, recursive theories that can interpret an elementary amount of arithmetic (*ZFC* has its own Rosser sentence). Given these

8 Something like this argument is endorsed in Putnam (1967a) but then mocked in Putnam (1967b).

9 The Rosser sentence R intuitively "says" that if R is provable, then there is a shorter proof of $\neg R$. Unlike the standard Gödel sentence, the negation of the Rosser sentence for a theory extending arithmetic can be shown to be independent of the theory without assuming that the theory is ω-consistent. See Rosser (1936).

10 Just to make sure that there isn't any confusion here: recall that INDETERMINACY applies to subject matters by way of a notion of determinate truth that applies to sentences concerning the subject matter.

11 There are explicit arguments for this, but in the background to such arguments is typically the thought that syntax is determinate, and so, since the canonical examples of undecidable *arithmetical* sentences (Gödel sentences, Rosser sentences, consistency sentences) all concern, in some sense, syntactic facts about provability, they must be fully determinate. However, it is worth noting that not all examples of arithmetical incompleteness are of this nature; see Paris and Harrington (1977) for an example with no *prima facie* connection to the syntax of arithmetic.

disanalogies (and others), it might be wondered how there could be any mystery over the differential respect with which these arguments are treated.[12]

We don't deny any of these differences or their potential importance for the philosophy of mathematics. However, they are not strictly relevant at this point: here we are only pointing out that independence from some canonical theory of sets or arithmetic, *by itself*, is not enough to establish that a claim is indeterminate. To establish the indeterminacy of set theory or arithmetic, neither the A-argument nor the B-argument is sufficient—at the very least, more must be said about the account of mathematical truth that is being assumed. In fact: to investigate the determinacy of mathematics, we need to explicitly investigate the metasemantics of our mathematical language.

3 The metasemantic challenge

On all but the most revisionary views, mathematical vocabulary is meaningful: it possesses semantic properties of various kinds. Whenever a linguistic item has semantic properties, we can ask: what *explains* the fact that it has the semantic properties that it does? This, we take it, is the central question of metasemantics. Metasemantics ought to be distinguished from, on the one hand, semantics proper, which concerns the semantic properties of particular linguistic items, and on the other hand, theorizing about the metaphysical nature of meaning (are meanings fregean thoughts, russellian propositions, or what?) Of course, this is not to say that metasemantics is irrelevant to either semantics proper or the metaphysics of meaning, just that clarity is served by distinguishing the enterprises.

Determinacy, as characterized in Sect. 1, is a semantic property. As a result, metasemantic questions about determinacy naturally arise. We can ask: given that some mathematical sentences are determinately true or determinately false, what explains that fact? This is the question that motivates the metasemantic challenge:

> The Metasemantic Challenge: Mathematical determinacy, to the extent that it arises, must be explained.

Let us make some brief clarificatory and motivational comments.[13]

Firstly, what is the force of the "must" that figures in the challenge? The idea is that, if you endorse the determinacy of mathematics, you are rationally required to explain how that determinacy arises. To the extent that no explanation can be provided, there is significant rational pressure to reject the view that led to the claim of determinacy in the first place.

Secondly, how can the challenge be motivated? The presence of semantic properties—such as determinacy—is a matter that calls out for explanation. It would

[12] Other possible disanalogies include (i) the extent of disagreement over proposed alternative axioms and related claims (see Clarke-Doane 2013) and (ii) the potentially different role that considerations concerning nonstandard models may legitimately play in indeterminacy arguments in set theory as opposed to arithmetic (see Gaifman 2004). The first type of disanalogy noted here may also be relevant to the plausibility of mathematical pluralism.

[13] Our challenge has been influenced by Putnam (1980) and the surrounding literature, especially Hartry Field's series of papers—(1994, 1998a, b)—on arithmetical and set-theoretic determinacy.

be philosophically abhorrent to have to acquiesce to a "brute" metasemantic account of determinacy, according to which facts about determinacy don't have any kind of explanation. The semantic properties of sentences are simply not the sort of things that we can comfortably take as explanatorily basic. Just as it would be extremely unappealing—an abandonment of any serious metasemantic theorizing—to claim that the fact that a particular word has the meaning it does is a matter of brute fact (not explained by, say the conventions governing its use, or causal chains connecting it with things in the world, or some other account of this kind), so too would it be unappealing to make the analogous claim for determinacy.[14] This is a domain where explanation is needed.

Of course, from thoughts like this it also follows that a correct metasemantic theory of mathematics should also explain any *indeterminacy* in our mathematical language, but indeterminacy is often quite easily explained. For instance, it is explained easily by the simplistic approach to mathematical truth sketched in Sect. 2—if the truth of a sentence in the language of arithmetic is explained by the sentence following from the axioms of the distinguished theory of arithmetic, *PA*, and if the falsity of a sentence is equivalent to the truth of its negation, then the independence of *PA*'s Rosser sentence *R* would explain the indeterminacy of *PA*.[15]

It might be objected that the metasemantic challenge is somehow illegitimate, on the grounds that it attempts to hold accounts of mathematics to unreasonably (perhaps even skeptically!) high explanatory standards. But that is not so. Although we've formulated it in terms of *mathematical* language, we take the spirit behind the metasemantic challenge to be fully general: wherever determinacy arises, it must be explained. For instance, someone who holds that discourse about medium-sized dry goods is fully determinate might give an explanation of how this is so in terms of the fact that we are engaged in (some suitable) causal contact with those objects. Obviously, this is the merest beginning of a sketch, the point is just this: requiring an explanation of the determinacy of some domain is not an extravagant or *ad hoc* demand that we are wheeling out to embarrass mathematics in particular; rather, it is an essential part of a satisfying overall metasemantic view.

A helpful analogy can be drawn between the metasemantic challenge and the *epistemological* challenge for mathematics made famous by Paul Benacerraf and Hartry Field.[16] Stated briefly, the epistemic challenge says that mathematical reliability, to the extent that it arises, must be explained.[17] The analogy should be clear. Both of these challenges proceed from the premise that certain putative facts—determinacy, reliability—require an explanation and cannot comfortably be postulated as brute.

[14] Here and throughout we assume that words and sentences are individuated syntactically. Of course the very same metasemantic questions will arise, in a slightly altered form, if linguistic items are individuated semantically.

[15] Of course, saying that indeterminacy is often easy to explain, metasemantically, is not the same thing as saying that the concept of indeterminacy, or the semantics of an indeterminacy-operator, is easy to explain.

[16] See Benacerraf (1973) and the introduction to Field (1989) for the most important contributions to the literature on epistemological arguments against realism. And see Warren (2016) for a recent generalization and defense of such arguments.

[17] Cf. Schechter (2010). In Warren and Waxman (Unpublished) we develop and endorse a generalized epistemological argument against realism along the lines suggested by Schechter.

Both of these challenges are manifestations of challenges that apply more generally (to *any* domain which is determinate/within which we are reliable). And neither of these challenges is objectionably skeptical: it is at least feasible to see how both can be surmounted in various non-mathematical domains.

Now, the epistemological challenge for mathematics is widely believed to have real bite. In particular, *realist* views, according to which mathematical claims are about mind-independent abstract objects, are thought by many to be incapable of meeting the epistemological challenge and hence undermined (or even refuted outright) on that basis. So the obvious question arises here: are there any views which are analogously undermined or refuted by the metasemantic challenge? To investigate that issue, we will need to turn to a discussion of the constraints on acceptable metasemantic theories.

4 Two constraints on an acceptable metasemantics

Like any science, metasemantics must be compatible with basic scientific naturalism; we believe that this gives rise to two constraints on acceptable metasemantics that are relevant to the metasemantic challenge. Here is the first:

> The Metaphysical Constraint: abstract objects cannot play an essential role in the metasemantics of mathematics.

Before motivating the metaphysical constraint, it will be helpful to distinguish between two grades of platonic involvement—two different levels of commitment that theorists might adopt with respect to mathematical objects. The *first grade* of platonic involvement involves acceptance of the existence of mind-independent mathematical objects. The *second grade* involves, in addition, taking mind-independent mathematical objects to play a serious explanatory role in our theorizing about non-mathematical domains—including, saliently for our purposes, metasemantics.[18]

The metaphysical constraint follows from the rejection of the second grade of platonic involvement. Of course, it also follows from rejecting the first grade, but this isn't required. Our strategy for defending the metaphysical constraint is to point out that the second grade of platonic involvement is in conflict with a plausible and undemanding form of scientific naturalism. Because we will be opposing the second and not the first grade of involvement, nothing in these arguments should be construed as requiring nominalism (more on this below).

There are many versions of naturalism, but nearly all of them entail that the physical realm is an explanatorily closed system, so that the proper explanation of some natural phenomenon cannot make essential appeal to abstract objects like numbers or sets. This is not, of course, to deny that mathematics can play an extremely important and useful role in scientific explanations; that would be hard to contest. But it is to deny that the role of mathematical objects in these explanations is essential. For instance, the fact that the laptop is exerting a force of 9.81 newtons on the desk is explicable in

[18] Our distinction between the first and second grades of platonic involvement is similar to Field's (1984) distinction between "moderate" and "heavy-duty" platonism, though we don't focus just on relations between abstract and physical objects but rather on the explanatory role played by the former.

terms of the fact that it it has a mass of 1 kg and it is being acted upon by a gravitational field of strength 9.81 m/s^2. But surely the *real numbers* 9.81 and 1 do not play any essential role in this explanation—the explanation would be just as good if it were translated into some different unit of measurement, for example.[19]

Some may think that we are thus committing ourselves to being able to formulate an entirely nominalistic theory of the world, but that is a bit too hasty. Even if we can't formulate an entirely nominalistic theory, it may be enough to point out that no particular type of mathematical objects are playing an essential role in scientific explanations, given the plethora of distinct, but still platonistic, explanations available to us. The lesson to draw from this multitude of equivalent descriptions may be that none of the theory-specific mathematical posits have any true *physical* significance. We take this to be a guiding posit of naturalism, but we do not claim there is a general way to transparently read off *what is physically significant* from a given scientific explanation or theory.[20]

Before proceeding to the second constraint, it is worth explaining in more detail why acceptance of the metaphysical constraint does not require nominalism. The thought that it does is based on the idea that if mathematical terms, such as the numeral "3", refer to mathematical objects, such as the number 3, then the constraint will be violated, since the proper metasemantic story of mathematics will need to vindicate these reference clauses, and doing so will, *eo ipso*, violate the metaphysical constraint. This is a tempting thought, but it is not correct. Reference clauses such as these will only attribute an essential explanatory role to mathematical objects if their truth cannot be accounted for in independent terms. But there are many platonistic proposals for accounting for mathematical reference that do not violate the metaphysical constraint. Let us briefly mention two.

Firstly, there are deflationary and disquotational accounts of reference, which come in various forms, all holding that reference is not an explanatorily robust property, but is instead explained or defined via all instances of the disquotational reference schema: "*a*" refers to *a*. It seems plausible to us that a metasemantics for mathematics that successfully incorporates a disquotational theory of reference has legitimately accounted for reference, without explanatorily substantive appeal to mathematical objects.[21] Secondly, accounts of reference in the Neo-Fregean tradition in the philosophy of mathematics embrace a sentential priority thesis, according to which facts involving sub-sentential semantic notions, like reference, are to be explained in terms of facts involving sentential semantic notions, like truth. This is then coupled with a view in which the truths of mathematics flow from implicit definitions that are themselves true by stipulation. When these two steps are combined, facts about reference to mathematical objects are explained using facts about stipulations concerning mathematical

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19 See Woods (2016) for some related thoughts.

20 This is important to note because the physical significance of some scientific theories—such as quantum mechanics—is hotly contested. For some relevant discussion of the quantum mechanical case, in the context of whether and how nominalist programs such as Field's (1980) could apply to quantum mechanics, see Malament (1982) and Balaguer (1996) for discussion.

21 See Leeds (1978) for relevant discussion.

sentences, and so the metaphysical constraint is not violated.[22] If successful, either of these approaches could be used to account for reference to mathematical objects without violating the metaphysical constraint.

According to the metaphysical constraint, the role played by mathematics in scientific explanations is *merely representational* —the underlying physical phenomena are explanatorily adequate, in themselves. Rejecting this, we think, entails a mysterious view of the world that is at odds with even the most benign forms of scientific naturalism.[23]

We don't think that the metaphysical constraint is particularly controversial, even amongst contemporary platonists, but the second of our constraints on an acceptable metasemantics is perhaps even less controversial:

> THE COGNITIVE CONSTRAINT: humans cannot be attributed non-computational causal powers

The cognitive constraint is justified by the modern scientific picture of human cognition. This picture takes human cognition to result from neural—and hence, mechanical—operations performed by the human brain. The general idea is nicely expressed by Vann McGee:

> Human beings are products of nature. They are finite systems whose behavioral responses to environmental stimuli are produced by the mechanical operation of natural forces. Thus, according to Church's Thesis, human behavior ought to be simulable by a Turing machine. This will hold even for idealized humans who never make mistakes and who are allowed unlimited time, patience, and memory.[24]

Church's thesis asserts that the informal notion of algorithmic computability extensionally coincides with the formal notion of Turing computability. Of course, it isn't essential that we use Turing computability as our formal model of computation: Turing computability itself is provably extensionally equivalent to recursive computability and every other formal model of computability that has ever been seriously proposed.[25] Church's thesis itself isn't open to a rigorous proof, since it involves the informal notion of algorithmic computability, but no serious threat to the thesis has ever been proposed.

Violations of the cognitive constraint must contravene either Church's thesis or the modern scientific picture of the human mind. As such, the cognitive constraint has considerable force.[26] To us, this seems entirely conclusive, but a few brave souls have rejected the constraint. Perhaps most prominently and relevantly, J.R. Lucas and Roger Penrose have, in effect, used the assumption of arithmetical determinacy to argue

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[22] For Neo-Fregeanism, see Wright (1983), Hale (1987), and the essays in Hale and Wright (2001).

[23] For this reason, we also accept a general version of the metaphysical constraint, applying to the metasemantics of all other domains. However, since our concern here is with mathematics alone, that is the focus of our attention.

[24] (McGee, 1991, p. 117).

[25] See any recursion theory textbook for details, e.g., Odifreddi (1989).

[26] It also has non-trivial repercussions; see McGee (1991).

against the cognitive constraint.[27] Lucas and Penrose perform a *modus tollens* where we perform a *modus ponens*. For our part, we think this to be extremely implausible: arithmetical determinacy is far less secure than either Church's thesis or the modern scientific picture of the human mind.

The metaphysical constraint and the cognitive constraint result from commitment to a fairly minimal form of scientific naturalism. We think that the relevant form of naturalism is accepted, either explicitly or implicitly, by the vast majority of scientists, philosophers, and educated laymen. Wide acceptance is no guarantee of correctness, of course, but rejecting the most basic and fundamental part of our modern scientific worldview is not something that should be done lightly, *even by philosophers*. The next section uses our constraints to provide an appealing and diagnostically useful, but perhaps not incontrovertible, argument against global mathematical determinacy.

5 An argument for indeterminacy?

To put things roughly: if our mathematical language is fully determinate, then that determinacy must arise either from the world or from our practice. Intuitively, the metaphysical constraint undercuts the possibility of using *the world* to explain mathematical determinacy, while the cognitive constraint undercuts the possibility of using *our practice* to explain mathematical determinacy.

It seems to us that these two options—explaining determinacy using the world or explaining it using our practice—are exhaustive. To some extent, the exhaustiveness might be seen as simply stipulative, but we don't want to put too much weight on this. If an explanation of determinacy was offered that didn't intuitively come from either the world or our practice, we would consider our claim of exhaustiveness undermined. We should also note that, to some extent, *our practice* plays a role in all putative explanations of determinacy. So to say that the world explains determinacy is really to say that the practice-independent world, in conjunction with our practice, accounts for determinacy. For this reason, we shouldn't be understood as claiming that the interaction of the world and our practice cannot be a major factor in explaining mathematical determinacy, to the extent that it exists. But as long as this is kept in mind, nothing is lost if we mainly restrict our focus to broadly world-based and broadly practice-based explanations of determinacy, respectively.

The metaphysical constraint undercuts the idea that determinacy arises from the world by blocking platonistic and heavily metaphysical explanations of mathematical determinacy. The most straightforward form such putative explanations take would involve using an explanatorily freestanding and fully determinate mathematical realm to account for the determinacy of our mathematical language. But this is clearly ruled out by the metaphysical constraint. And the cognitive constraint undercuts the idea that determinacy arises from our practice by blocking all attempts to see our practice as encoding determinate and thus non-recursive mathematical theories. The argument

[27] The primary sources here are Lucas (1961), Penrose (1989, 1994); helpful responses include Benacerraf (1967), Boolos (1990) and Shapiro (1998, 2003).

for this is straightforward: the cognitive constraint seems to guarantee that our total practice is algorithmic and thus, by Church's thesis, recursive, but Gödel's incompleteness theorems entail that no consistent, recursive theory of arithmetic is negation complete. From this it seems to follow that whatever theory of arithmetic is encoded in our practice, it cannot be fully determinate.

All of this leads to a powerful argument for mathematical indeterminacy (with indeterminacy understood as in Sect. 1):

1 Mathematical determinacy arises, if at all, either from the world or from our practice
2 If the metaphysical constraint is true, then mathematical determinacy does not arise from the world
3 If the cognitive constraint is true, then mathematical determinacy does not arise from our practice
4 The metaphysical constraint is true
5 The cognitive constraint is true
6 *So*: mathematical determinacy does not arise, that is, our mathematical practice is indeterminate (1–5)

We think this, or something very much like it, is the best argument that can be made for the indeterminacy of mathematics. Just as importantly, this argument, even if deemed unsound, provides a valuable diagnostic tool for pinning down the often elusive commitments of those who accept mathematical determinacy.

Before considering possible responses, it is worth stressing that this argument is—ostensibly at least—still sound when the word "mathematical" is replaced with "arithmetical" throughout. In other words, while the argument puts pressure on the assumption of global mathematical determinacy, a variant of the argument can be used to put pressure, locally, on arithmetical determinacy even at the level of first-order arithmetic. This is because *true arithmetic*, the negation complete theory containing all true sentences in the language of first-order arithmetic, is non-recursive.

With this in mind, let's consider the space of alternatives generated by this argument, starting at the end. In Sect. 4 we discussed and endorsed both the metaphysical constraint and the cognitive constraint; for the reasons mentioned there, we don't think that the argument can plausibly be blocked either by rejecting premise 4 or premise 5. As already discussed, anyone who rejects either of these constraints, in effect rejects basic scientific naturalism in order to account for mathematical determinacy. It is preferable, and far less radical, to block the argument by rejecting one of the first three premises instead. In the remainder of this section, we will briefly set out the most plausible options for doing this, first restricting our focus to arithmetic, and then briefly considering possible extensions to set theory. As will become apparent, it is far from obvious that *any* of the strategies for blocking the argument and securing mathematical determinacy are ultimately sustainable.

In some sense, it is a vague matter whether a particular strategy will count as rejecting premise 1 as opposed to premises 2 or 3. This is because it is, in some sense, vague whether or not something counts as part of "the world" or as part of "our practice". Nevertheless, most strategies can naturally be seen as opting either to reject premise 2 or to reject premise 3. Likewise, some of the strategies we consider could be thought of as undermining premise 4 or 5, but as just noted, we will not consider them in this capacity, given the discussion of the previous section.

So, continuing in reverse order, there are at least three broad types of strategies for rejecting premise 3 that have been pursued in the literature.[28]

The first attributes to us an ability to follow infinitary rules, like Hilbert's ω-rule:

$$(\omega) \quad \frac{\varphi(0), \varphi(1), \varphi(2), \ldots}{\forall x \varphi(x)}$$

Adjoining the ω-rule to even very weak arithmetical theories results in true arithmetic. For this reason, if our practice included the ω-rule, then we would have a plausible explanation of (first-order) arithmetical determinacy. However, it doesn't seem like we could follow the ω-rule without violating the cognitive constraint. This is widely accepted, but some philosophers have denied it. For example, Carnap says:

> In my opinion, however, there is nothing to prevent the practical application of such a rule.[29]

Unfortunately, it isn't at all clear what Carnap meant by this. Some later philosophers have thought that the ω-rule provides a useful idealization of our practice, and others have helped themselves to the rule without explaining how it is we can follow it, but the overwhelming majority of philosophers have thought that we can't follow the ω-rule without violating the cognitive constraint.[30]

The second strategy for rejecting premise 3 appeals to a version of mathematical induction stronger than the standard first-order induction schema, leading to a theory of arithmetic for which categoricity can be proved. For example, it is well-known that second-order Peano arithmetic, which includes the following axiom of mathematical induction:

$$\textbf{SO Induction}: \quad \forall X\Big(\big(X0 \wedge \forall x(Xx \rightarrow Xs(x))\big) \rightarrow \forall x Xx\Big)$$

is categorical. As standardly conceived, this result requires the second-order quantifiers to range over *all* numerical properties in extension or (equivalently) *all* sets of natural numbers. But this strategy just pushes the question of determinacy back a step: an explanation is needed of how *second-order quantification* could be determinate; and here we are in a familiar space of alternatives. One could of course appeal to entities "out there" that serve as the value of second-order variables, and attempt to explain determinacy that way; but that would amount to a violation of the metaphysical constraint. Or one could seek to ground the determinacy of second-order quantifiers in our practice; but the standard semantics is computationally rich, so it is hard to see

[28] One approach we will not discuss attempts to use Tennenbaum's theorem to argue that arithmetic must be categorical and hence determinate—see Dean (2002) and Button and Smith (2011) offer a critique.

[29] Quoted from (Carnap, 1934, p. 173).

[30] See Chalmers (2012) and Weir (2010) for use of the ω-rule as an idealization and Horwich (1998) for unexplained appeal to it.

how we can appeal to it without violating the cognitive constraint.[31] For this reason a mere move to second-order logic is powerless to explain arithmetical determinacy.[32]

A related approach that arguably doesn't face the same problem uses the standard first-order induction schema but interprets the schematic letters in an *open-ended* fashion:

$$\textbf{Induction} : ((\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(s(x))) \rightarrow \forall y \varphi(y))$$

The idea behind an open-ended understanding of induction is that rather than being limited to predicates definable in our current language, induction continues to apply as we add further and further expressive resources to our arithmetical language. There are numerous ways of developing this idea, but according to all of them, either a version of the categoricity result can be proven, or a philosophical argument for determinacy can be launched. This strategy for securing determinacy has been pursued in various forms by Shaughan Lavine, Vann McGee, and Charles Parsons.[33] However, many critics have questioned whether open-endedness does any better than a move to second-order logic in accounting for determinacy. The critics point out that, an open-ended understanding of induction is naturally seen as equivalent to the Π^1_1-fragment of second-order logic, and it is hard to see how a sub-theory of second-order logic could be better at securing determinacy than full second-order logic itself.[34]

The third and final strategy for rejecting premise 3 claims that we, in some fashion or other, can grasp the standard model of arithmetic via a cognitive act of imagination or some similar faculty. This is a bit imprecise, but this type of strategy attempts to vindicate the intuitive thought that we can *conceive* of the standard model of arithmetic, in some fashion. A Kantian version of this approach might explain this ability as an offshoot of our temporal intuition, or something similar, but non-Kantian versions of the strategy are also possible. Obviously, it is unclear what to say about this general approach without an explicit proposal on the table, but it is very hard to see how *any* version of this strategy could avoid violating either the metaphysical constraint or the cognitive constraint. If the cognitive act of imagination involved is perception-like, in that it involves accessing, in some quasi-perceptual fashion, an explanatorily independent mathematical object (the standard model of arithmetic), then it would seem to violate the metaphysical constraint. But if it does not, then it would seem to violate the cognitive constraint, since it would then require that somehow, the standard model of arithmetic can be located entirely "in the head", so to speak.

Let us now discuss a strategy that rejects premise 2, which claimed that the metaphysical constraint blocks all attempts to use the world to explain mathematical determinacy. Such an approach has been developed by Hartry Field in a series of papers.[35] Field's idea is that, in effect, premise 2 is false because we can secure the

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[31] See Shapiro (1991) for details of standard and other semantics for second-order logic.

[32] This type of criticism was pioneered by Weston (1976).

[33] See Lavine (Unpublished), McGee (1997) and Parsons (2001).

[34] See the discussion in Field (2001).

[35] Field (1994, 1998a, b).

determinacy of our arithmetical vocabulary using the physical world. Let S be the theory consisting of our set theory, pure and applied, together with our total physical theory and define a predicate $\Phi(Z)$ of S as follows:

> Z is a set of events which (i) has an earliest and a latest member, and (ii) is such that any two members of Z occur at least one second apart

Let's also make the following cosmological assumptions:

1. Time is infinite in extent (there is no finite bound on the size of sets satisfying Φ)
2. Time is Archimedean (any set satisfying Φ will have only finitely many members)

In S we will be able to derive a principle concerning our "finitely many"-quantifier, $\mathcal{F}$, linking there being finitely many Ψ's to there being a one-one mapping from the Ψ's into some set of events Z satisfying Φ. If we assume that the physical vocabulary in our background theory S has no non-standard interpretations (for instance, interpretations where "event" is assigned an extension that includes non-events), then we can also rule out non-standard interpretations of $\mathcal{F}$, and then use the determinacy of $\mathcal{F}$ to induce determinacy in arithmetic by adding an axiom stating that every number has only finitely many predecessors. In effect, Field uses the physical world to explain mathematical determinacy, but this seems to get the intuitive conceptual priority here backwards, and Field himself is somewhat ambivalent about his proposal, writing:

> It might be thought objectionable to use physical hypotheses to secure the determinacy of mathematical concepts like finiteness. I sympathize—it's just that I don't know any other way to secure determinacy.[36]

Charles Parsons has pressed an objection along these lines in criticizing Field's approach:

> I find it hard to see how someone could accept that assumption who does not already accept some hypothesis that rules out nonstandard models as unintended on mathematical grounds. If our powers of mathematical concept formation are not sufficient why should our powers of physical concept formation do any better? We are not talking about events in a common-sense context, but rather in the context of a physical theory developed in tandem with sophisticated mathematics.[37]

We sympathize with Parsons's worry, but won't further discuss the matter here. In any case, Field's approach—or some variant of it—seems to us the only somewhat plausible option for using the world to secure determinacy while respecting the metaphysical constraint.

As we hope to have made clear, all of these approaches face significant *prima facie* problems. It is at best highly unclear whether any of them can be worked out in a satisfactory fashion. And even if one of them can, none of these strategies obviously

[36] (Field, 1998b, p. 342); a similar quote from page 418 of his earlier (1994) is still ambivalent, but slightly less skeptical of his own approach and slightly more skeptical of alternatives: "I am sure that some will feel that making the determinateness of the notion of finite depend upon cosmology is unsatisfactory; perhaps, but I do not see how anything *other* than cosmology has a *chance* of making it determinate".

[37] (Parsons, 2001, p. 22).

generalize in a satisfying fashion beyond the arithmetical domain. In fact, most of them seem entirely hopeless as strategies for securing *set-theoretic* determinacy. For instance: there is no generalization of the ω-rule to the standard set theoretic case that humans could plausibly follow. And a second-order or open-ended understanding of the schemata in *ZFC* (replacement and separation) will at best attain *quasi*-categoricity, in the sense that any two models will either be isomorphic or one will be isomorphic to a proper initial segment of the other.[38] The prospects for the grasp of models strategy are even more bleak, for if it is implausible that we can grasp a model of arithmetic via an act of cognitive imagination, it must be several orders of magnitude more implausible that we could grasp the standard universe of set theory via such an act. Finally, Field's approach, likewise, would require both implausibly rich powers of physical concept formation and implausibly strong cosmological assumptions in order to be applied to the set-theoretic case.

To be clear, we are not admitting that any of these strategies succeeds in securing the determinacy of arithmetic in the face of our argument. Instead, we are merely pointing out that even *if* any of these strategies were to succeed in the case of arithmetic, none of them obviously extend to set theory, and so we'd still be left with substantial mathematical indeterminacy. Exactly how much indeterminacy will depend on the particular strategy. The extent of determinacy secured by a Field-style approach, for example, will depend upon how much determinacy can be offloaded onto the physical world.[39] In any case though, all of these strategies are hopeless if intended to account for full mathematical determinacy and this hopelessness is, perhaps, one reason that the literature has tended to opt for a quite different approach. Let us now turn to this approach.

6 New axioms?

From what we can tell, the most popular approach to resolving particular instances of putative mathematical indeterminacy is to search for new axioms. To illustrate: if we are concerned with settling *CH*, since we know that *CH* is not settled by standard set theory (*ZFC*), we should—according to this approach—look for some new set of axioms, *A*, such that adding *A* to standard set theory settles *CH*. Of course, the approach involves not simply the search for *any* axiom that settles *CH*—*CH* itself would be such an axiom, and naïve comprehension, since inconsistent, settles *every* set-theoretic question at once—instead the idea is to *justify* or *motivate* acceptance of

[38] See Zermelo (1930) and Isaacson (2011). McGee (1997) proves a full categoricity result for open-ended set theory, but only by both (i) adding an urelemente set axiom and (ii) making the crucial assumption that the quantifiers in each theory range over the very same domain.

[39] A referee for this journal has noted to us that it might be possible to develop certain conditional limits on Field's approach in moving from first to second-order arithmetic. The idea being that even if Field's approach secures determinacy for first-order arithmetic, given the cognitive constraint, this determinacy couldn't plausibly be extended to (for example) all claims concerning Π^1_2-sets. One interesting point here is that this criticism can be made while allowing that the interaction of the world and our practice extends determinacy beyond what is accounted for by either factor, all on its own.

the new axioms. This admits of a number of further distinctions and qualifications (is the motivation extrinsic or intrinsic?, etc.) but let us press on to the general case.[40]

Taken on its own terms, this procedure is fine if our concern is only to settle a particular mathematical claim such as *CH*. However, we submit that as an approach to accounting for determinate truth, that is, for global mathematical determinacy, even for basic arithmetic, this approach is completely hopeless. The reason for this is simple: add any new axioms you like to *PA* in order to wind up with arithmetical theory $PA+A$; now if determinate arithmetical truth in our language is somehow accounted for by $PA+A$, this must be either because $PA + A$ is itself a determinate theory or because our practice, with axioms $PA+A$, together with the world, accounts for determinacy. But, by the argument of the previous section, if the former, then the theory must be non-recursive and thus the cognitive constraint is violated, and if the latter, then the metaphysical constraint is violated. Of course, this doesn't mean that $PA+A$ cannot itself be extended by adding further well motivated arithmetical axioms. The point is just this: at any stage in the process of adding new axioms, the same dilemma will arise. In short: the search for new axioms, while perhaps able to resolve *some* undecidable sentences, is entirely helpless in accounting for full determinacy *even for first-order arithmetic*. If arithmetical truth is fully determinate, as most of us suppose, then the metasemantic challenge cannot, even in principle, be met by searching for new axioms.

Philosophically speaking, the most sophisticated versions of the new axioms strategy are heavily influenced by a more or less Quinean approach to epistemology that emphasizes theoretical virtues.[41] The basic idea is to appeal to general principles of theory choice in motivating new axioms. On such a view, a natural account of, say, determinate arithmetical truth sees it as extensionally equivalent to entailment from the *best* theory of arithmetic, even though this best theory is not yet, and may never be, a theory that we have actually formulated. Arguably, various complications aside, something like this Quine-inspired approach is accepted by many prominent contemporary philosophers of mathematics, including Peter Koellner and Penelope Maddy.[42]

It might be thought that, in appealing to the "best" theory of arithmetic in order to explain arithmetical truth, this approach avoids our dilemma; the thought being that, since we need not currently accept the best theory of arithmetic, there is no violation of either the metaphysical or cognitive constraints. It would be analogous to a theist accounting for truth in our arithmetical language by appeal to the language of God—presumably, this can be done without assuming that our meager minds have fully grasped God's arithmetical theory. However, as far as we can see, the dilemma arises even for the quinean. On this picture, our principles of theory choice single out the best theory. But if principles that are somehow encoded in our global practice of theory choice, even if only implicitly, single out a determinate theory of arithmetic, then we have just moved the mystery back up a step. If our principles of theory choice single out a determinate theory of arithmetic, it must be either by way of the world

40 See Gödel (1964) for some relevant further discussion.

41 See the final section of Quine (1951).

42 See Koellner (2010) and Maddy (1997, 2007). We are unsure whether either of these philosophers thinks that full and complete determinacy can be attained in this fashion.

or our practice, and so we've landed ourselves back in the dialectical situation from Sect. 5, on a slightly different level.[43] And if there is some third path, between the world and our practice, then this must be explained, and we seriously doubt that this can be done. For these reasons, as far as we can see, the excursus through principles of theory choice has not bought us much.

7 Coda

So where do things stand? In particular, does the metasemantic challenge lead to a watertight argument against mathematical determinacy even at the level of basic arithmetic? We believe that we've said enough to establish that the argument is certainly interesting and worth taking very seriously, but is it *sound*? Maybe. We have already said why we think the metaphysical and cognitive constraints must be accepted. If we are right about this, the soundness of the argument will depend upon whether an approach to securing determinacy can be made out without violating either of our constraints. Just how much determinacy can be secured without violating our constraints is unclear, but, as we've already seen, there are serious reasons to doubt the determinacy of even basic arithmetic, to say nothing of set theory or the whole of mathematics. Of course, more work remains to be done.[44]

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[43] The talk of "singling out" here concerns our *practice* singling out a *determinate theory*, rather than our *theory* singling out (determinately) certain *mathematical objects*. That is: we are still talking about determinate truth rather than determinate reference to mathematical objects.

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