Challenges to Objectivity in Mathematics

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1 Objectivity and Determinacy

No one knows whether there are infinitely many pairs of prime numbers differing by 2; this is perhaps the most famous unsolved conjecture in number theory. But although the truth-value of the conjecture isn't known, it seems hard to deny that it *has* a determinate truth-value one way or another. Intuitively, we could discover it if we could simply run through the natural numbers, checking one by one. Of course, we cannot actually run through infinitely many cases; but this looks like a problem of our epistemic limitations, nothing more.[1]

Contrast this with the Continuum Hypothesis (CH). Cantor famously showed that the cardinality of the real numbers is strictly larger than $\aleph_0$, the cardinality of the natural numbers, and conjectured that it is $\aleph_1$, the next largest infinite cardinality. But CH is independent of the axioms of ZFC, the quasi-official foundation for mathematics; indeed, ZFC places few constraints on the cardinality of the continuum.[2] We do not currently know whether CH is true or false. But many suspect something deeper: there is no fact of the matter to know. CH concerns a portion of mathematical reality – if it is that – where many have less of a clear intuitive conception and less conviction that every question is settled. But of course, attitudes differ. Some philosophers and mathematicians hold that CH has a definite answer, perhaps even one we may hope to discover by establishing new axioms.[3] Others argue that there is indeed no fact of the matter. For instance, on Hamkins' 'multiverse' view, there are many equally legitimate set-theoretic universes (roughly: models of ZFC), some in which CH is true and others

[1] As Russell (1936) put it, a medical impossibility, not a mathematical one. Part of the literature on 'supertasks' discusses whether running through the cases may even be *physically* possible (Hogarth 1992; Earman and Norton 1993). For connections to the determinacy of mathematics, see Berry (2014) and Warren and Waxman (2020b). There may also be routes to mathematical knowledge that do not go via proof (Paseau 2015); I set these aside here.

[2] Gödel 1940; Cohen 1963; Easton 1970.

[3] Woodin 2001a,b; Koellner 2009; Kant forthcoming.

false, with none privileged in a way that settles the matter.[4]

These two cases bring out a cluster of issues regarding objectivity that have been a central preoccupation of the philosophy of mathematics, largely independent of more familiar ontological questions about mathematical existence. A famous statement of this perspective was attributed to Kreisel:

> The point is not the existence of mathematical objects, but the objectivity of mathematical truth.[5]

Objectivity is a slippery notion, and I will not try to define it. But as it figures in these debates it has a reasonably stable core theoretical role. To call a mathematical claim or question objective is to endorse its *factuality*: there is a fact of the matter about it; the question is settled, one way or another. Objectivity and its absence are typically taken to have epistemic consequences: for instance someone who accepts the wrong answer to an objective question is making a genuine cognitive error; and there is little point inquiring after the answer to a question one suspects has none. Objectivity is not meant to be *merely* an epistemic notion; these epistemic features are supposed to be downstream from a more fundamental unsettledness.

There are at least two ways to flesh out a notion with this theoretical role. A *metaphysical* conception concerns how things stand in mathematical reality itself; failures of objectivity consist in the facts themselves being 'gappy' or indeterminate.[6] On the other conception, the main question is whether the meaning or content of our mathematical vocabulary and concepts is sufficient to fully pin down a subject-matter in a way that settles all questions. The conception is thus *metasemantic*, concerning what fixes content, rather than semantic, concerning what the content is. This will be the focus of the chapter: it gives rise to the central challenge we will consider.

Return to the twin prime conjecture. At first glance, it concerns a unique mathematical structure – the natural numbers – and it has a determinate truth-value, known or not. Generalizing: say that a portion of mathematical language enjoys *uniqueness* if it picks out a unique structure (up to isomorphism), and *determinacy* if every sentence formulated in it is determinately true or determinately false. The two notions are easily run together, and it is natural to assume that uniqueness yields determinacy: if our language singles out a unique structure, every sentence receives a truth-value in it. One recurring lesson is that this transition is much less innocent than it looks.

The chapter is organized around a central challenge – some might regard it as sceptical in nature – to the effect that mathematics simply cannot be determinate. It is

[4] Hamkins 2012; see also Martin 2001.

[5] Dummett 1978, p. 228, though the attribution to Kreisel is fraught.

[6] See e.g. Barnes and Williams 2011; Bacon 2018; Goodsell 2022.

closely related to a line of argument descending from Skolem and most commonly associated with Putnam (1980), problematizing the idea that large parts of mathematics have 'intended interpretations'.[7] But, as I will explain, I think the more compelling version of the challenge relies on independence results, not model-theoretic results – Gödel, rather than Löwenheim–Skolem. Part of what makes the challenge so troubling is that it threatens the determinacy not just of higher mathematics, but of as basic and intuitively clear a part of mathematics as arithmetic. I will take arithmetic as my main example in what follows. If the challenge succeeds there, it succeeds nearly everywhere. However, as we go, I will also note how matters stand for set theory, where if anything the questions have more bite. The challenge is developed in §2. Responses occupy §§3–6.

2 The Challenge

This section sets out what I think is the strongest form of metasemantic challenge to the objectivity of mathematics. Hardly any philosophers accept the conclusion; but there is significant disagreement about which premise should be rejected.

In the background of the challenge is a certain naturalistically motivated picture of the human mind, which constrains the metasemantic resources available for explaining mathematical content.[8] Mathematical objects are causally inert, so causal interaction with them cannot fix reference or other semantic facts. One might posit a special faculty – 'mathematical intuition' or the like – which connects us with the mathematical realm; but such faculties sit poorly with a naturalistic conception of creatures like us, which holds out little hope of vindicating them.[9] Following Putnam (1980), call a view **moderate** if it respects this conception – as opposed to an *immoderate* view that credits us with some special faculty of access to the mathematical realm. Moderation severely limits metasemantics: it rules out any quasi-causal mechanism by which content is fixed. Postulating mathematical objects is no cheap way out: their existence fails to settle any metasemantic questions. If, as moderates hold, there is no reference by

[7] Cf. Benacerraf 1973; Field 1994.

[8] 'Naturalism' is an overburdened word, so let me briefly clarify some senses that motivate the challenge. One is that we are natural creatures whose cognition is a natural process. Another is that meaning is a natural phenomenon, so the determinacy of mathematical language must be explicable. (There is no presupposition of a 'first-philosophy' standpoint here.) The challenge is neutral on 'methodological naturalism' in the sense of Paseau and Marfori (2025); nothing in it is *mathematically revisionary* (although it challenges a widespread philosophical view *about* mathematics).

[9] The most prominent non-causal alternative is Lewisian reference magnetism Lewis 1983, 1984: meaning is a function of use and 'eligibility'. The standard model 'wins' solely by being more eligible than its rivals, even though use does nothing to single it out. Moderates will typically view this as postulating mysterious metasemantics – primitive eligibility facts about causally inert structure that secure determinacy by fiat. See Button and Walsh 2018, §§2.4–2.5, Warren 2024.

acquaintance, content must be fixed in some other way. Many moderates take that way to be, specifically, our *inferential practice* – the pure mathematical axioms we accept together with the inferential resources we use in drawing out their consequences.[10] Let us call this claim **Practice-Dependence**.

For the challenge to apply, we must move from mathematical practice – a kind of activity that creatures like us participate in – to formal theories which faithfully capture it. This move needs non-trivial idealization, from a messy human activity to a clean formal theory.[11] One important source of resistance, which we will discuss extensively, is that formal theories have the wrong *shape* to faithfully regiment mathematical practice. But it is worth first appreciating why, from a moderate perspective, it is natural to think they do.

The standard reasoning here appeals to the computational theory of mind, in a suitably general form: human cognition is fundamentally a kind of computation. If so, then our cognitive capacities are bounded, roughly speaking, by what a Turing machine can do.[12] If we conceive of mathematical practice as drawing out consequences from axioms (understood in a neutral sense, as inferential starting points), then both the axioms and the consequence relation are subject to computational constraints. First, the set of axioms of mathematical practice must presumably be recursively enumerable (r.e.), as the outputs of a Turing machine must be. And second, the inferential practice of drawing consequences from those axioms must be both finitary – each inference essentially appeals to only finitely many premises – and effective – so that whether a proposed inferential step is valid is effectively checkable.

From these constraints it follows that the set of sentences obtainable from the accepted axioms by accepted inferential resources is recursively enumerable. Call this **Finitude**. It is precisely what the limitative results need to bite.

One form of the challenge can now be stated. A familiar form of Gödel's incompleteness theorem tells us that any consistent, recursively axiomatized first-order theory containing minimal arithmetic is incomplete, i.e. there are sentences in its language that it cannot prove or refute.[13] But first-orderness is not essential: what is needed is that *the set of theorems of the theory in question is recursively enumerable*. Say that a body of facts settles a sentence when it metasemantically suffices for its truth/falsity. The challenge can now be stated explicitly:

[10] Cf. Button and Walsh 2018, p. 152: 'the theories *themselves* precisely pin down the isomorphism types'. See also Putnam 1980; Parsons 2008; Soysal 2020.

[11] Though see Koellner 2018b.

[12] Note that the claim applies equally to 'neural network' models as to classical computational models, and that it assumes a kind of physical Church–Turing thesis. The most famous denial that human cognition is computationally bounded comes from the Lucas–Penrose argument, which I briefly discuss in §5.

[13] Smith (2007, Ch. 22).

Moderation. An arithmetical sentence is determinate only if naturalistically acceptable facts settle it as true or settle it as false.

Practice-Dependence. In pure arithmetic, the only naturalistically acceptable facts capable of settling sentences are facts about inferential practice: they settle a sentence only if it (or its negation) can be inferred from the axioms.

Finitude. The set of sentences that can be so inferred is recursively enumerable.

Gödel. No consistent, recursively enumerable set of sentences containing minimal arithmetic includes, for every arithmetical sentence, either it or its negation.[14]

Conclusion. Some arithmetical sentences are indeterminate.

The argument is valid, given the (typically uncontested) assumptions that mathematical practice is consistent and includes minimal arithmetic.

Note that the challenge does not presuppose Kripkensteinian scepticism about rule-following. It does not doubt that there are facts about the rules we follow; it needs only that the ones we do follow are recursive, and so fail to decide some sentences.

The presentation so far has a proof-theoretic flavour: mathematical practice fails to decide certain sentences, and indeterminacy threatens. There is another, more semantic, way of looking at matters. Pretend for a moment that mathematical practice is best captured by a first-order theory. Now consider its models (thought of as 'possible interpretations'). From the fact that there are sentences undecided by the theory, and the completeness theorem for first-order logic, it follows that there are models of the theory which *disagree about the truth-values of sentences in the language of arithmetic*. Put another way: mathematical practice fails to single out a unique interpretation, or even a class of interpretations that settle each question the same way.[15]

Why not present the argument in the usual, Skolemite way, following Putnam in appealing to the Löwenheim–Skolem theorem? It tells us that any first-order theory with infinite models has models of different infinite cardinalities. But while these models arguably undermine *uniqueness*, and arguably the determinacy of *reference*, they do nothing to undermine determinacy of *truth-value*: the (non-isomorphic) models that Löwenheim–Skolem gives us need not disagree about any sentence's truth-value. A clear way to see this is by noting that the Löwenheim–Skolem theorem applies equally

[14]See Smith (2007) for more on 'minimal arithmetic'; it suffices that the theory interprets Robinson Arithmetic Q. Strictly speaking, this statement expresses Rosser's refinement of Gödel's theorem.

[15]This coheres with the supervaluationist picture of determinacy as truth on every admissible interpretation. The connection with vagueness returns in §3.3.

well to $T(\mathbb{N})$, the set of all first-order sentences true in the standard model of arithmetic – a negation-complete but non-recursively-enumerable theory.

The assumption that our theories are first-order is not essential to show that there are divergent interpretations. Suppose we have a theory formalized some other way. There will be many ways to generalize the notion of model to the new resources. But the moderate perspective constrains which of these can be regarded as genuine interpretations of the theory. A class of models gives rise to a notion of semantic consequence: truth-preservation over the class.[16] Now, for moderates, the only resources capable of settling mathematical content come from mathematical practice. So: a class of models can only be regarded as a genuine interpretation of our theories if the consequence relation it yields coincides with inferential consequence. Call a class of models with this property *faithful*. If inferential consequence outruns semantic consequence, the 'interpretations' leave open possibilities foreclosed by practice; and if vice versa, the models build in more content than the practice contains. The point generalizes: any *faithful* class of interpretations will, as in the first-order case, reflect what happens at the level of inferential consequence. Just as before, incompleteness yields divergent interpretations.

The same reflections address the intuitively appealing thought that surely, somehow, we can 'grasp' or 'see' the *standard* or intended model of arithmetic. As Feferman (2011, pp. 14–15) puts it, our conception of the natural numbers 'is so clear' that there should be no question as to arithmetical claims having a determinate truth-value.

But again the issue is *how* this is supposed to be possible. Moderation rules out anything like *seeing* the natural numbers themselves or their structure; if we single it out, it must be via our practice. But, if the practice satisfies the apparently well-motivated constraints, it simply can do no such thing. The 'standard' model is just one admissible interpretation of our practice, among others that materially disagree with it.

3 Determinacy via Categoricity

This section discusses responses which argue that mathematical practice can single out a unique interpretation (or isomorphism class of interpretations) via categoricity results: first external, model-theoretic categoricity (§3.1), then 'internal' categoricity (§3.2). §3.3 develops a problem common to all.

[16] I'll understand 'class of models' here to include structures across all signatures, what model-theorists sometimes call a 'logic'.

3.1 Second-Order Categoricity

An influential line of thought attempts to resist the challenge by arguing that mathematical practice is best formalized by a *second-order* theory.[17] The headline result is that second-order arithmetic (Dedekind 1888) is categorical in the sense that all of its models – with one large caveat – are isomorphic to one another. A direct corollary, seemingly highly relevant to the issue of determinacy, is what has been called an intolerance result: any two models of second-order arithmetic – again with the same caveat – must agree on the truth-values of all sentences in the relevant language. The caveat is that the result holds only provided that consideration is restricted to *full* models, in which the second-order quantifiers range over the entire powerset of the first-order domain. So, the response goes, our practice *does* single out a unique isomorphism class of interpretations, agreeing on the truth-values of all sentences, and determinacy is achieved.

However, this response is typically regarded as unsuccessful.[18] Full models are a subclass of the broader class of Henkin models, in which the second-order quantifiers range over some (possibly proper) subset of the powerset of the first-order domain.[19] Full models and Henkin models give different notions of semantic consequence. Recursive proof systems, generalizing those for first-order logic, are sound and complete for Henkin consequence; by contrast no recursive calculus is complete for full consequence. Henkin consequence behaves much more like first-order consequence than full consequence – in particular, Gödel's theorem applies: for any consistent recursively axiomatized theory containing minimal arithmetic, there are sentences on which its Henkin models disagree.

In the terms of §2, the second-order categoricity response amounts to a denial of Finitude: it claims that our practice somehow tracks full consequence and thereby settles a non-recursively-enumerable set of sentences. The problem is that the restriction to full models is unmotivated by anything in our practice: the class of full models is not faithful, in the sense discussed above. Now the class of Henkin models at least has the right shape to be faithful.[20] But then the argument of the challenge bites once again: there will be *mutually divergent* Henkin models of the axioms, and no recursive consequence relation will be able to rule all such models out.[21]

[17] For arguments that second-order languages better capture various aspects of informal mathematics, see Kreisel (1967) and Shapiro (1991).

[18] For the locus classicus, see Weston 1976. See also Putnam 1980; Field 2001, postscript to Ch. 12; Parsons 2008, §48; Warren 2020, p. 244; Picollo and Waxman 2023.

[19] See Shapiro 1991, Chs. 3–4.

[20] It depends on the details of inferential practice. If it is exhausted by a calculus like those above, for which Henkin models are sound, then the class of Henkin models is faithful.

[21] Assuming consistency. Note that for any two divergent Henkin models, one can be excluded by moving to an expanded theory (adding a sentence true of the 'standard' structure that fails in it), or by adding a new rule of inference to the calculus; but if the axioms are r.e. and the notion of consequence

A defender of full models might insist that the restriction needs no further motivation: when we use second-order quantifiers, 'all' means all; and only full models respect that constraint (Shapiro 1991, Ch. 8). But this simply *presupposes* determinacy rather than explains it (Weston 1976). True, Henkin models 'leave out' some subsets. But the entire problem is to show how our practice excludes them. That 'all' means all does little more to show second-order quantification determinate than 'bald' means bald shows 'bald' precise. One might instead argue that Henkin models can be ruled out by ascending to a set-theoretic metatheory in which the distinction between full and Henkin models can be explicitly defined. The problem is that the same issue recurs at the level of the metatheory (Parsons 2008, p. 274) – the metatheoretic distinction between full and Henkin models is, in Putnam's (1980) sense, 'just more theory'. If the metatheory is first-order, it has non-isomorphic interpretations, each containing internal models of arithmetic it regards as full, but which are not isomorphic to one another.[22] If the set-theoretic metatheory is instead second-order – a case to which I'll return in a moment – the question simply re-arises: why are *its* Henkin interpretations excluded? The restriction to full models, which the ascent was supposed to legitimize, is at every stage presupposed.[23]

How do matters stand with set theory? Second-order ZF is not categorical but quasi-categorical (Zermelo 1930): given two full models, one is isomorphic to an initial segment of the other. A naive reading: the axioms pin down everything but the 'height' of the universe. Kreisel (1967) famously used this observation to argue that CH, despite its independence from first-order ZFC, is decided by full second-order consequence, since CH concerns only a fixed low level of the hierarchy ($V_{\omega+2}$). But, again, any attempt to draw philosophical conclusions about *determinacy* from this runs into the same problem of justifying the restriction to full models.[24]

3.2 Internal Categoricity

The categoricity results we have just considered are *external*, concerning the *model-theoretic* interpretations that theories can have; and as we saw, this model-theoretic lens generated problems. I will now turn to some attempts to achieve determinacy by appealing to *internal* categoricity arguments, not stated in model-theoretic terms at all.[25]

recursive, there will nevertheless be further divergent Henkin models compatible with everything the expanded system proves.

[22] That is, there are models $\mathcal{M}_1$ and $\mathcal{M}_2$ and non-isomorphic models ω_1 and ω_2 such that ω_1 satisfies the definition of a full model of arithmetic in $\mathcal{M}_1$ and similarly for ω_2 in $\mathcal{M}_2$.

[23] Putnam 1980; Parsons 2008; Warren and Waxman 2020a; Picollo and Waxman 2023.

[24] See Kreisel (1967) and Weston (1976) for the original debate.

[25] The term 'internal categoricity' originates with Walmsley (2002).

The first is due to Parsons (2008, §49), who proves an 'essentially first-order' version of Dedekind's categoricity result and draws philosophical conclusions from it. The technical situation is this. Suppose we start with two copies of the axioms of first-order arithmetic, stated in disjoint signatures, each of whose induction schemas is understood 'open-endedly' (i.e. ranging over *all* vocabulary of the language, including possible future expansions). Then, assuming the legitimacy of defining functions by recursion, one can define a function between the two copies of the numbers that sends zero to zero and successors to successors.[26] It can be further proven, from the combined axioms, that this mapping is an *internal isomorphism*: it is bijective and provably preserves arithmetical structure.[27] Call this an *Internal Categoricity* result. As a corollary, if φ is a sentence in the signature of one of the copies of arithmetic, and φ^* is the analogous sentence in the other signature, it is provable from the combined axioms that $\varphi \leftrightarrow \varphi^*$. Call this an *Internal Intolerance* result.

What do these results show? At least that it is *intra-personally* incoherent to hold two disparate conceptions of arithmetic side by side. Parsons, however, wishes to draw a further, *inter-personal* lesson: that Internal Categoricity rules out that two different speakers have disparate conceptions. Suppose Alpha and Beta each accept a copy of the axioms, and each interprets the other by translation. The idea is to use Internal Categoricity to show that each must regard the other's numbers as isomorphic to their own, and the other's sentences as agreeing in truth-value with their own. Whether this works is delicate – it depends on exactly how each regards the other's translations as behaving – and on careful analysis the argument turns out to assume what it was meant to show: at a crucial point, Alpha must presuppose that Beta's numbers are not non-isomorphic to his own.[28] So the inter-personal application of Internal Categoricity is dubious. There is also a deeper problem: even if the strategy succeeds, it would still fail to establish determinacy of truth-value. I defer discussion for a moment, since it afflicts both Parsons' view and the next.

In recent work Button and Walsh attempt to deploy a different, though related, internal categoricity result to argue for determinacy.[29] Their 'internalism' (taking inspiration from Putnam's internal realism) makes two main conceptual moves. First, they adopt second-order logic, but insist on understanding it not through a model-theoretic lens but as part of an object-language whose meaning is given by a programme of use. Second, all talk of 'structures' and 'interpretations' of a theory is carried out within the

[26]The legitimacy of recursive definition here has raised some controversy: see Maddy and Väänänen (2023), Fischer and Zicchetti (2023).

[27]Parsons 2008, §§47–49.

[28]Briefly: the argument assumes that there is a term *in Beta's language* that Alpha must translate as 'number'. But this rules out by fiat that Beta's numbers are 'non-standard' from Alpha's perspective. See Field 2001, postscript to Ch. 12, Picollo and Waxman 2025.

[29]Button and Walsh 2016, 2018; Button 2022.

second-order object language itself. For instance, the claim that there is a unique arithmetical structure is formalized, not as a model-theoretic claim, but as a second-order sentence:

$$\forall N_1 z_1 s_1\, \forall N_2 z_2 s_2 \left[\big(\mathrm{PA}(N_1, z_1, s_1) \wedge \mathrm{PA}(N_2, z_2, s_2)\big) \rightarrow \exists R\ \mathrm{Iso}_R(N_1, z_1, s_1; N_2, z_2, s_2) \right]$$

understood as saying that any two satisfiers of the axioms are isomorphic. This is the form of Internal Categoricity to which they appeal.[30] The crucial point is that Internal Categoricity is not a metatheoretic claim about classes of models, but a sentence of the object-language itself and – by a variant of Dedekind's original argument – a *theorem*.[31] So, internalists conclude, we can show that there is a unique arithmetical structure, in a way that sidesteps the question of full vs Henkin consequence altogether, and with it the complications concerning faithful classes of models.

Internalists also have a distinctive response to concerns about determinacy. A fairly immediate corollary of Internal Categoricity is what Button and Walsh call the Intolerance schema: $\mathtt{t}\varphi \vee \mathtt{t}\neg\varphi$ for each arithmetical sentence φ, where $\mathtt{t}$ abbreviates the second-order claim that φ holds in every structure satisfying the arithmetical axioms.[32] This is simply a theorem, again in fact provable in pure second-order logic. Its alleged philosophical significance comes from the further move of *interpreting* $\mathtt{t}\varphi$ as 'it is determinately true that φ'. So interpreted, the schema – all of whose instances are theorems – shows that each sentence is either determinately true or determinately false. So, the internalist claims, determinacy too has been established by proof alone. In terms of the challenge in §2, this strategy amounts to a rejection of Moderation – not by way of bringing in non-naturalistic resources, but by denying that our practice need settle a sentence in order for it to be determinate. I will shortly argue, in §3.3, that this cannot be sustained.

In many ways, the internalist's categoricity result is an object-language, second-order version of Parsons'. And, similarly, it seems to achieve something in the intra-personal case: one cannot simultaneously accept two disparate conceptions of arithmetic (extending the second-order axioms). The inter-personal case has not been discussed in any detail. The only way I can see to conclude that two speakers must regard each other's numbers as isomorphic to their own is to assume that each interprets the other's second-order quantifiers homophonically – which turns on controversial issues about quantification we cannot pursue here. My suspicion is that this amounts to assuming the very sameness of conception the argument was to establish.

[30] See also the distinction between general and particular structures in Isaacson (2011).

[31] No arithmetical premises are needed: the conditional is a theorem of pure (impredicative) second-order logic, provable in any standard proof system.

[32] $\mathtt{t}\varphi := \forall N \forall z \forall s\, (\mathrm{PA}(N, z, s) \rightarrow \varphi^{N,z,s})$, where $\varphi^{N,z,s}$ is the relativization of φ.

Turning briefly to set theory, the situation is analogous with interesting complications. An internal quasi-categoricity theorem can be proven in second-order logic.[33] Full Intolerance does not follow unless the height of the universe is fixed (by e.g. an additional axiom saying that no inaccessible cardinals exist). Button and Walsh's own development (2018, Ch. 11) uses a Scott–Potter-style stage theory rather than ZF; adding the postulate that the pure sets are equinumerous with the universe yields full categoricity.[34] Notably, since the theory includes no axiom of infinity, the result is a fully internally categorical set theory with finite models – an illustration of how little internal categoricity constrains external models.

3.3 Wide- vs Narrow-Scope Uniqueness and Intolerance

Even waiving all this, a more powerful objection afflicts every argument from categoricity we have seen.[35] Let me present it for internalism first, and then draw a general lesson.

Determinacy is a metasemantic notion – to say that φ is determinate is to say that its truth-value is settled by the relevant metasemantic facts. And, especially for moderates, such facts cannot comfortably be taken as brute. Now consider a sentence like the Rosser sentence R, which the internalist's theory neither proves nor refutes.[36] By Intolerance, and the interpretation of t, the internalist holds that it is determinately true that R or determinately false that R. Either disjunct faces an insurmountable explanatory demand: to explain how the determinacy arises. For the internalist's metasemantic resources are limited to the consequences of their recursive theory; but since both R and $\neg R$ are independent of that theory, no explanation of the determinacy of either can be forthcoming. The right way out, it seems to me, is to give up on the interpretation of t as determinacy; and this collapses the whole strategy.

More generally: categoricity approaches derive (perhaps within a metatheory) a categoricity claim – that any two interpretations of arithmetic are isomorphic – and an intolerance claim – that no two can disagree about any sentence's truth-value. But for all that, determinacy has not been established. It is helpful to think about it as a matter of scope. There is a 'wide-scope' uniqueness claim: it is determinate that – settled by our practice that – any two arithmetical structures are isomorphic. But it does not follow that there is some structure such that it is determinate that any interpretation of arithmetic is isomorphic to *it*. In parallel, there is a 'wide-scope' intolerance claim:

[33] See Väänänen and Wang 2015; Maddy and Väänänen 2023.

[34] Cf. McGee (1997).

[35] Field 2001, Warren 2020, Ch. 10.

[36] The Rosser sentence 'says' of itself that if it is provable, there is a shorter proof of its negation. By a tweak on Gödel's argument, the Rosser sentence is neither provable nor refutable (assuming only consistency) (Rosser 1936).

it is determinate that either φ is true in all arithmetical structures or $\neg\varphi$ is true in all arithmetical structures. But it does not follow that either it is determinate that φ is true in all arithmetical structures or that it is determinate that $\neg\varphi$ is true in all arithmetical structures:

$$\text{(Wide Uniqueness)} \quad \mathrm{Det}\,\forall S_1 \forall S_2 \left(\mathrm{Arith}(S_1) \wedge \mathrm{Arith}(S_2) \to S_1 \cong S_2\right)$$

$$\text{(Narrow Uniqueness)} \quad \exists S\,\mathrm{Det}\,\forall S' \,(\mathrm{Arith}(S') \to S' \cong S)$$

$$\text{(Wide Intolerance)} \quad \mathrm{Det}\,(\mathrm{t}\varphi \vee \mathrm{t}\neg\varphi)$$

$$\text{(Narrow Intolerance)} \quad \mathrm{Det}\,(\mathrm{t}\varphi) \vee \mathrm{Det}\,(\mathrm{t}\neg\varphi)$$

where the structure-variables may be read model-theoretically or internally, as one prefers. The theorems deliver (Wide Uniqueness) and (Wide Intolerance); determinacy of reference and truth-value require (Narrow Uniqueness) and (Narrow Intolerance); and the step from one to the other is a scope fallacy familiar from vagueness: it is determinate that Harry is either bald or not, but it does not follow that he is determinately bald or determinately not.[37]

Again, undecidable sentences are the crux – the ghost of Gödel is still with us. Nothing in mathematical practice, moderately conceived, could explain which of them or their negations is determinately true. Categoricity arguments give us the wide-scope claims, but determinacy requires the narrow-scope claims. The next two sections consider strategies that attempt to establish them directly, by rejecting Practice-Dependence (§4) and Finitude (§5).

4 Determinacy via 'Narrow-Scope' Uniqueness

I turn now to strategies that aim to establish 'narrow-scope' uniqueness directly: some interpretation is such that it (or anything isomorphic to it) is the unique adequate interpretation of arithmetic. Views of this kind have the right shape to answer the challenge: if the standard model is singled out, determinacy of truth-value arguably follows. In the terms of §2, the strategies of this section reject Practice-Dependence: each brings in facts beyond pure inferential practice to do content-fixing work. §4.1 considers views appealing to facts about our broader dispositions;[38] §4.2, facts about the

[37] That 'determinately' does not distribute over disjunction is a point on which semantic theories of vagueness agree (Fine 1975; McGee and McLaughlin 1995; Keefe 2000).

[38] Some proponents might prefer to say that inferential practice, properly construed, is more expansive than moderates would allow (in effect rejecting Finitude rather than Practice-Dependence). Nothing really turns on this; the important questions are whether the additional resources are legitimate, and whether they achieve determinacy.

physical world, actual or merely possible.

Mathematical theories like set theory and arithmetic, at least when presented in their typical first-order forms, are schematic: for instance Peano Arithmetic includes the induction schema:

$$\text{(Ind)} \qquad \big(\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n{+}1))\big) \to \forall n\,\varphi(n)$$

There are two ways to understand such schemas. One is that they allow substitution-instances from a language *fixed in advance*. In normal contexts, arithmetical induction applies to sentences in the language of arithmetic. But another, much-discussed, way to understand schemas is as *open-ended*, holding with equal force for *expansions* of our language.[39] As McGee (1997, p. 36) puts it, our understanding of the natural numbers ensures that 'we don't have to reassess the validity of mathematical induction when we expand our inventory of theoretical concepts'. A recurring thought in the literature is that our open-ended acceptance of induction over these possible expansions is metasemantically relevant in constraining the interpretation of arithmetical language and sufficient to single out a unique structure.

4.1 Adding a 'Standardness' Predicate

One approach, hinted at in McGee (1997, 2000), is that open-ended induction can be used to rule out any non-standard models of arithmetic as follows. Suppose that, in some expansion of our language we have a predicate 'St' that holds of all and only the *standard* natural numbers, obtained from 0 by finitely many iterations of the successor operation. Then by our open-ended acceptance of induction, we are committed to

$$(*) \qquad \big(St(0) \wedge \forall n\,(St(n) \to St(n+1))\big) \to \forall n\, St(n)$$

And this rules out any model $\mathcal{M}$ containing 'non-standard' numbers: any such $\mathcal{M}$ will have $\mathcal{M} \vDash St(0)$ and $\mathcal{M} \vDash \forall n(St(n) \to St(n+1))$ but $\mathcal{M} \vDash \neg\forall n St(n)$, contradicting $(*)$.[40]

This argument seems superficially plausible. But, as Field (2001, pp. 344–5) pointed out, it begs the question with the assumption that we can expand our language with a predicate that, 'by some magic', picks out the 'standard' numbers. As he puts it,

[39] There is a restriction to 'well-behaved' expansions, excluding e.g. vague or malfunctioning vocabulary; see Warren 2020, p. 10.II.

[40] Every non-standard model contains an initial segment isomorphic to the standard numbers (Kaye 1991).

'the only possible relevance of schematic induction is to allow you to carry postulated future magic over to the present; and future magic is no less mysterious than present magic'.

A recent tweak on McGee's proposal is due to Murzi and Topey (2021). Their idea is that open-endedness of induction is grounded in our dispositions to infer in accordance with it, and that these dispositions extend into metaphysically and logically possible scenarios – even, potentially, impossible ones. Some such scenarios include 'angels who are in quasi-perceptual contact with abstracta and who thereby can introduce into our language a predicate the extension of which is the set of [standard] natural numbers' (Murzi and Topey 2021, p. 3415). This is arguably an improvement on a bare appeal to future magic. But it does not obviously address the fundamental challenge: to show how determinacy could arise on naturalistic metasemantic assumptions. Even if, in some distant metaphysical possibility, we commune with angels, it is unclear why facts about what we are disposed to accept *there* bear on what we, here and now, mean.[41]

The open-endedness strategy has a set-theoretic analogue. McGee proves that any two models of an open-ended second-order set theory (with an axiom ensuring the urelements form a set) agree on their pure sets, yielding full (not quasi) categoricity. But the theorem requires the quantifiers to range over absolutely everything, including all sets. The natural concern is whether assuming the determinacy of unrestricted quantification, including over a domain of mathematical entities, is any more secure than the determinacy of mathematical vocabulary.[42]

4.2 Cosmological Hypotheses

I'll turn now to a strategy proposed by Field, who attempts to use *cosmological hypotheses* to rule out non-standard models. Here is the basic idea. Let S be a theory that includes applied set theory, together with physical vocabulary, in which a predicate Φ can be defined so that it applies to a set of events Z when (i) Z has an earliest and a latest member, and (ii) any two members of Z occur at least one second apart.

Now make the following cosmological assumptions:

1. Time is infinite in extent (i.e. there is no finite bound on the size of sets satisfying Φ).

[41] See Picollo (2026). Murzi and Topey also offer an answer to an analogous challenge for logic, drawing on Bonnay and Westerståhl (2016). For an invariance-based route to arithmetical determinacy in the same spirit, see Speitel (2024). A determinacy-theorist might also appeal to Williamson (2016), who argues that any mathematical truth is *absolutely provable* in that some possible agent could safely adopt it as an axiom. But the sentences such agents accept have no evident bearing on what *our* practice settles, which remains recursively enumerable; and the argument could bear on *our* sentences only if such axioms were determinately translatable into our own language – which §3.2 gives reason to doubt.

[42] McGee 1997; on absolutely unrestricted quantification, see Rayo and Uzquiano (2006).

2. Time is Archimedean (i.e. any set satisfying Φ has only finitely many members).

Then in S we will be able to define a 'finitely many' quantifier, where there are finitely many Ψ if there is a bijection between the Ψ and some set Z such that $\Phi(Z)$. The crucial metasemantic assumption is that *the physical vocabulary* in S is fully determinate (e.g. has no non-standard interpretations where 'event' is assigned an extension that includes non-events).[43] Then non-standard models can be ruled out: we can define 'finite number' in terms of the finiteness quantifier, and show via open-ended induction that 'all numbers are finite numbers' is true in any admissible model $\mathcal{M}$. Thus any $\mathcal{M}$ that assigns a non-standard extension to 'natural number' must also assign a non-standard extension to some physical vocabulary. So, Field concludes, determinacy has been established.

A counterintuitive consequence of this (ingenious) response is that it makes the determinacy and content of even pure arithmetic depend on contingent empirical assumptions. Indeed this is the heart of its response to the challenge: rejecting Practice-Dependence and allowing facts beyond practice alone to play a role in settling arithmetical truth. As Field (2001, p. 341) says, 'I sympathize – it's just that I don't know any other way'.

Two other criticisms are worth mentioning, though neither clearly succeeds. First, Parsons objects to the determinacy of the physical vocabulary on the grounds that notions like 'event' are defined in physical theories, themselves developed in tandem with sophisticated mathematics.[44] This raises large issues, but all the strategy needs is determinate reference to a physical ω-sequence. Its members could be far more mundane than events; if the world cooperates, even rocks or planets. There is no obvious reason why determinate reference to these must be hostage to sophisticated mathematics; the ancient Greeks could refer to planets as well as we can. Second, Bueno (2005) and Putnam (2016) have objected that Field's approach embodies a kind of circularity because it *uses* the notion of finiteness in a way that presupposes its determinacy – a form of objection worth taking seriously, since, as we have seen, it afflicts many other proposals. But Field's point is that the cosmological assumptions (plus determinacy of physical vocabulary) impose *external* constraints that rule out non-standard models; the determinacy of the assumptions plays no role.[45]

Before moving on, consider a version of this proposal due to Berry (2021). Her idea is that *merely physically possible* ω-sequences suffice, assuming the determinacy

[43] Of course, we could have defined an analogous quantifier in arithmetic plus a small amount of set theory ('the Ψ are in bijection with an initial segment of the natural numbers'); the obvious concern is that *that* quantifier is no more determinate than the mathematics in terms of which it's defined.

[44] Parsons 2008, p. 290.

[45] The Bueno–Putnam objection is a descendant of Putnam's 'just more theory' move (§3.1). The disanalogy: a metatheory is just more practice – whereas Field's constraint is external.

of the relevant modal vocabulary. Suppose that any infinite sequence of coin-flips is physically possible; then there will be some sequence whose heads-outcomes form an ω-sequence. As before, this can be used to argue that any model $\mathcal{M}$ with non-standard natural numbers will also have a non-standard interpretation of 'heads', 'coin-flip', or physical modality – and can thus be ruled out, by hypothesis. This approach avoids empirical hypotheses about the actual world. But it does so by assuming determinate reference not just to actual physical objects – for which externalist metasemantic resources are available to explain determinacy – but also to objects in counterfactual or merely possible situations. As Warren (2020, pp. 253–4) notes, it is unclear that the same externalist resources allow us to refer to non-actual coin-flips. And if our reference to them is mediated only by *description*, then any indeterminacy in that description will carry over – a serious challenge to the proposal.

Finally, note that no analogous strategy holds out much hope for set theory. There is an analogue of the open-endedness move: the Axiom of Separation is a schema, which, read open-endedly, would exclude 'width-deficient' interpretations of set theory – just as ($*$) excludes non-standard models of arithmetic – provided the language could be expanded by predicates determinately picking out the missing subsets. The difficulty lies in finding any plausible physical structure to serve as a witness. Even if the world supplies an ω-sequence, it is hard to see what would play the role of arbitrary subsets, let alone of the higher ranks.[46]

5 Determinacy via Infinitary Reasoning

The previous strategies sought to establish determinacy via a unique structure. I turn to one that denies Finitude directly: the consequence relation implicit in our practice is genuinely infinitary, deciding every arithmetical sentence. The proposal, made in passing by Carnap (1937) and recently revived in Warren (2020, 2021), is that we follow the ω-rule:

$$\frac{\varphi(0) \qquad \varphi(1) \qquad \varphi(2) \qquad \cdots}{\forall x\, \varphi(x)}\ (\omega)$$

It turns out that adding the ω-rule to a weak arithmetical base theory (e.g. Robinson Arithmetic Q) yields a negation-complete theory. So if it could be shown that we really do follow the rule, determinacy would be secured. Could finite beings like us somehow follow it?

[46]If spacetime is a continuum and we can determinately speak of arbitrary fusions of points – in effect the second-order spacetime structure Field's (1980) nominalist programme once assumed – that *may* give us determinacy for arbitrary sets of reals. (Though see Malament (1982).) The major issue here is that 'arbitrary region' is no more clearly determinate than 'arbitrary set'.

The obvious objection is that we could never write down a proof that applies the rule – there is not enough paper in the world. But the real question involves not proof, but *inference* – a causal transition between mental states – and Warren has offered an ingenious argument that there is nothing in principle impossible about this. Here he appeals to a *dispositionalist* view: having a belief involves suitable dispositions to assent or act, rather than a particular mental representation. On this view, being in an infinite belief-state is relatively anodyne: most of us believe, of each natural number n, that it is either even or odd. Why can't such belief-states be the starting points of inferences?

Even granting all this, establishing that we follow the ω-rule requires some footwork. Suppose we really do seem to make some ω-inferences. If our disposition to accept the premises results from believing some finite generalization entailing them, then the case is better described as a *finitary* inference from that generalization to the conclusion.[47] So the infinite belief-state had better be *irreducible*: not grounded in any such finite belief. But it is hard to see how finite agents like us could have the disposition to accept infinitely many structurally similar claims without some finite generalization more immediately in mind. (The even-or-odd beliefs, after all, are fully explained by the single belief that *every* number is even or odd.)

Warren's most ambitious move is a case he thinks *forces* the hypothesis that we reason in an irreducibly infinitary way: a putative supertask computer, capable of checking infinitely many numerical cases in finite time. In it, he claims, we – but not an interlocutor who has a similar arithmetical practice *minus* the ω-rule – would come to accept arithmetical generalizations on the basis of the computer's output. Much of the strength of his argument turns on the details of the case. Several commentators have argued that a finitary-reasoning hypothesis accounts for the data in a simpler way; but discussion continues.[48]

Compare the most famous denial of Finitude: the Lucas–Penrose argument, which uses Gödel's theorem to argue that human cognition outruns any Turing machine, since we can 'see' that the Gödel sentence of any theory we accept is true.[49] The argument's flaws are well-documented.[50] The trouble, from the present perspective, is that no mechanism is offered to explain *how* the 'seeing' works. Warren's proposal is more interesting because it at least tries to offer a psychologically and philosophically acceptable mechanism, whatever its other flaws.

[47] Nyseth 2021. Note that if the inferences we make are all finitary in this sense, the original challenge is back. Compare Paseau and Griffiths (2021, §4), who argue that English consequence is not compact, while granting that our grasp of infinitary arguments proceeds by finite reasoning. Their claim concerns implication, not inference – so its bearing on Finitude turns on the issue under discussion: whether we can follow the corresponding rule.

[48] Nyseth 2021; Topey 2025; Blue 2025.

[49] Lucas 1961; Penrose 1989.

[50] Benacerraf 1967; Shapiro 1998; Koellner 2018a,b.

Finally, note that the strategy emphatically does not carry over to set theory. Any analogous rule would presumably require *class-many* premises, and a language with *class-many* names, one for each set. While such rules may be interesting to study from a logical perspective, there is no real prospect that we or any creatures remotely like us actually follow them.[51]

6 Consistency Sentences

One issue remains, which has lurked beneath the surface: the undecidable arithmetical sentences delivered by Gödel's theorem express the consistency of formal theories. This might be thought to generate an objection as follows. Whether a contradiction can be derived from a formal theory is a matter of syntax – perhaps about abstract syntactic objects, or perhaps about which concrete inscriptions can be written down – and surely claims of *this* kind must be determinate. But now if consistency facts are determinate, and if mathematical sentences like the Gödel sentence G express consistency facts, then must not such sentences be determinate too? An argument of this form has been pressed by Putnam and, following him, Koellner.[52] It threatens to undermine the central challenge, at least for *Gödelian* incompleteness (though it has no analogue for *set-theoretic* incompleteness).

There is a potential lacuna in the argument: why think that the *bridge-principle*, the biconditional between the syntactic consistency claim and the arithmetical claim, is itself determinately true?[53] Arithmetical claims like Con(PA) express consistency-claims via Gödel coding: they say that no number is the code of a proof of $\perp$. But this is equivalent to genuine consistency only on a *standard* interpretation; on interpretations with *non-standard* 'numbers', these would code up 'non-standard proofs' that correspond to no finite string of symbols. So even if one accepts the determinacy of syntax, the thought runs, concerns about the determinacy of arithmetic give reason to doubt the determinacy of the bridge principles.

But this is not ultimately a stable response.[54] If we formalize our *syntactic* commitments, we plausibly arrive at a theory that is a notational variant of arithmetic. In particular, it sustains syntactic induction: if some property holds of the empty string, and if given that it holds of a string it also holds of strings obtained by appending one more symbol, it holds of all strings. On mild further assumptions it can be *proven* that the bridge-principle between arithmetic and syntax holds (in effect via an appli-

[51] Fujimoto (2012).

[52] Putnam (1983), Koellner (2009). For recent discussion of similar issues, see Clarke-Doane (2022) and Berry (2023).

[53] Warren 2015. I am transposing his discussion into the key of determinacy.

[54] Picollo and Waxman (2025).

cation of Internal Categoricity); and so it seems hard to deny that such principles are determinate.

What I think this shows is that it is incoherent to regard arithmetic and syntax as coming apart in the way the response requires. Now, this does not necessarily rescue the Putnam–Koellner argument. If syntax is determinate, then arithmetic is determinate too. But it is worth thinking harder about the grounds for the antecedent here.

Suppose first that syntax is a theory of a distinctive kind of abstracta – string-types and the like. Then, it seems to me, the determinacy of syntax faces precisely the same challenges as the determinacy of arithmetic. The dialectic is familiar: the putative entities are causally inert, the metasemantic resources for explaining determinacy seem to be limited to our best theories about them, which themselves are constrained by our computational limits... and off we go. Of course I agree that the determinacy of syntax is highly plausible. But so too is the determinacy of arithmetic! Reasserting its plausibility does not, unfortunately, resolve the challenge of explaining how it is even possible.

Alternatively, we might conceive of syntax as a theory of possible concrete inscriptions. If we go this route, we have in effect a version of the *modalized* Field–Berry strategy, where the ω-sequence doing the metasemantic work is not a possible sequence of coin-flips, but a sequence of possible inscriptions. Again, although I do not have a knockdown argument against that approach, it inherits the same challenges: to establish determinate reference not to actual objects but to merely possible ones, and to the modal vocabulary in which they are described.

7 Conclusion

The challenge of §2 is pressing, and although I wish I had a knockdown argument that one of its premises is false, I do not. As we have seen, all of the attempts to meet it have issues of their own. For all that has been said, its conclusion may simply be true: perhaps mathematics, even arithmetic, is indeterminate. But note one striking feature of the landscape we have surveyed. Setting aside views that postulate mysterious metasemantic faculties and deny moderation outright, all of the responses with a serious prospect of success exploit something peculiar to the natural numbers: an ω-sequence that is or could be physically realized, or that arises from syntax itself, or our capacity to follow the ω-rule. But nothing remotely analogous is on offer to explain the determinacy of set theory. Those who endorse it, I think, have their work cut out.[55]

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